ANALYSIS

OF

ROTARY MOTION,

AS APPLIED TO

THE GYROSCOPE.

BY

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PREFACE.

The apparatus discussed here under the name Gyroscope was exhibited by Professor Walter R. Johnson, of the University of Pennsylvania, in 1831. It was then called the Rotascope, but it excited but little interest.

Professor Foucault, of France, brought it forward in 1855, and employed it as a means of making the earth's rotation visible to the eye. Since that time some form of Gyroscope (the name given it by Foucault) has formed a part of the philosophical apparatus for schools.

For some time the impression prevailed in the popular mind that the phenomena exhibited by the apparatus could not be explained by natural laws. This idea was perhaps strengthened by the name applied to it by Professor Olmstead, who called it "The Mechanical Paradox."

The following analytical exposition of

the motions of the Gyroscope was written by General (then Major) Barnard in 1858, for the *Journal of Education.* It was immediately reprinted in pamphlet form and was eagerly sought for by students of Analytical Mechanics. It yet remains the best treatise on this interesting apparatus.

As the former editions were long since exhausted, while the demand for the essay continued, it was considered advisable to republish it in its original form, first as a Magazine article and then as a volume of the Science Series.

G. W. P.

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After reading most of the popular explanations of the above phenomenon given in our scientific and other publications, I have found none altogether satisfactory. While, with more or less success, they expose the more obvious features of the phenomenon and find in the force of gravity an efficient cause of horizontal motion, they usually end in destroying the foundation on which their theory is built, and leave an effect to exist *without a cause;* a horizontal motion of the revolving disk about the point of support is supposed to be accounted for, while the descending motion, which is the first and direct effect of gravity (and without which no horizontal motion can take place), is ignored or supposed to be entirely eliminated. Indeed, it is gravely

stated as a distinguishing peculiarity of rotary motion, that, while gravity acting upon a non-rotating body causes it to descend vertically, the same force acting upon a rotary body causes it to *move horizontally*. *A tendency to descend* is supposed to produce the effect of an *actual descent;* as if, in mechanics, a mere tendency to motion ever produced any effect whatever without that motion actually taking place.

Whatever "mystification" there may be in analysis—however it may hide its results under symbols unintelligible save to the initiated, it is most certain that the greater portion of the physical phenomena of the universe are utterly beyond the grasp of the human mind without its aid. The mind can—indeed it *must*—search out the inducing causes, bring them together and adjust them to each other, each in its proper relation to the rest; but farther than that (at least in complicated phenomena) unaided, it cannot go. It cannot *follow* these causes in all their various actions and reactions and at a

given instant of time bring forth the results.

This, analysis alone can do. *After* it has accomplished this, it indeed usually furnishes a clue by which to trace how the workings of known mechanical laws have conspired to produce these results. This clue I now propose to find in the analysis of rotary motion as applied to the gyroscope.

The analysis I shall present, so far as determining the equations of motions is concerned, is mainly derived from the works of Poisson (vide "Journal de l'Ecole Polytech." vol. XVI—Traité de Mécanique, vol. II, p. 162). Following his steps and arriving at his analytical results, I propose to develop fully their meaning, and to show that they are expressions not merely of a visible phenommenon, but that they contain within themselves the sole clue to its explanation; while they dispel all that is mysterious or paradoxical, and in reducing it to merely a "particular case" of the laws of "rotary

motion," throw much light upon the significance and working of those laws.

Although not unfamiliar to mathematicians, it may not be uninteresting to those who have not time to go through the long preliminary study necessary to enable them to take up with Poisson this special investigation, or whose studies in mechanics have led them no farther than to the general equations of "rotary motion," found in text books, to show how the particular equations of the gyroscopic motion may be deduced.

In so doing I shall closely follow him; making, however, some few modifications for the sake of brevity and of avoiding the use of numerous auxiliary quantities not necessary to the limited scope of this investigation.

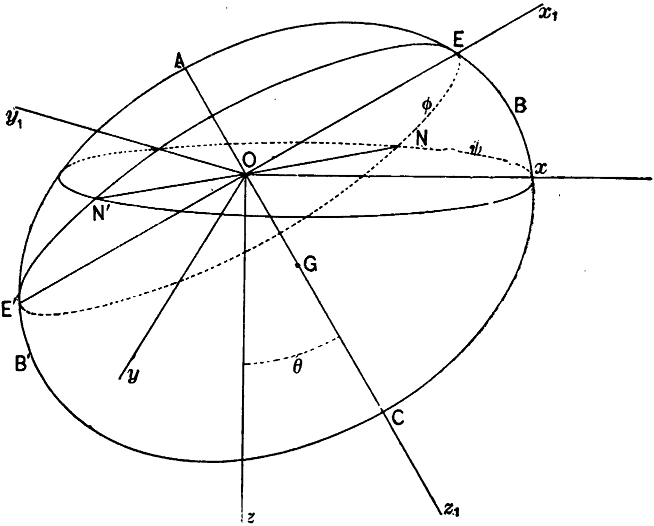
The general equations of rotary motion are (see Prof. Bartlett's "Analytical Mechanics," Equations (228), p. 170):

$$\left.\begin{aligned} C\frac{dv_z}{dt}+v_xv_y(B-A)&=L_1 \\ B\frac{dv_y}{dt}+v_xv_z(A-C)&=M_1 \\ A\frac{dv_x}{dt}+v_yv_z(C-B)&=N_1 \end{aligned}\right\} \quad (1)$$

In the above expressions the rotating body (of any shape) ABCD, Fig. 1, is supposed retained by the *fixed* point within or without its mass) O. Ox, Oy and Oz are the three co-ordinate axes, *fixed in space*, to which the motion of the body is referred. Ox_1, Oy_1, Oz_1, are the three *principal axes* belonging to the point O, and which, of course, partake of the body's motion. The position of the body at any instant of time is determined by those of the moving axes.

A, B. and C express the several "moments of inertia" of the mass with reference, respectively, to the three principal axes Ox_1 Oy_1 Oz_1; N_1, M_1 and L_1 are the moments of the *accelerating forces*, and v_x, v_y, v_z, the *components of rotary*

Fig. 1



velocity, all taken with reference to these same axes.

Like lineal velocities, velocities of rotation may be decomposed—that is, a rotation about any single axis may be considered as the resultant of components about other axes (which may always be reduced to three rectangular ones): and by this means, about whatever axis the body, at the instant we consider, may be revolving, its actual velocity and axis are determined by a knowledge of its components v_x, v_y, v_z, about the principal axes $Ox_1 Oy_1 Oz_1$, these components being, as with lineal velocities, equal to the resultant velocity multiplied by the cosine of the angles their several rectangular axes make with the resultant axis.

As the true axis and rotary velocity may continually vary, so the components v_x, v_y, v_z, in equations (1) are variable functions of the time.

For the purpose of determining the axes Ox_1, Oy_1 and Oz_1, with reference to the (fixed in space) axes Ox, Oy, Oz, three auxiliary angles are used.

If we suppose the moving plane of x_1 y_1, at the instant considered, to intersect the fixed plane of xy in the line NN′ and call the angle $x\text{ON}=\psi$, and the angle between the planes xy and x_1y_1 (or the angle $z\text{O}z_1)=\theta$, and the angle $\text{NO}x_1=\varphi$, (in the figure, these three angles are supposed *acute* at the instant taken) these three angles will determine the positions of the axes $\text{O}x_1$, $\text{O}y_1$, $\text{O}z_1$, (and hence of the body) at any instant, and will themselves be functions of the time; and the rotary velocities v_x, v_y, v_z, may be expressed in terms of them and of their differential coefficients.

For this purpose, and for use hereafter in our analysis, it is necessary to know the values, in terms of φ, θ and ψ, of the cosines of the angles made by the axes $\text{O}x_1$, $\text{O}y_1$ and $\text{O}z_1$ with the fixed axes $\text{O}z$ and $\text{O}y$.

These values are shown to be (vide Bartlett's Mech., p. 172)

$$\cos.\ x_1\text{O}z = -\sin.\ \theta \sin.\ \varphi$$

$$\cos.\ y_1\text{O}z = -\sin.\ \theta \cos.\ \varphi$$

$$\cos.\ z_1Oz = \cos.\ \theta$$

$$\cos.\ x_1Oy = \cos.\ \theta \cos.\ \psi \sin.\ \varphi - \sin \psi \cos.\ \varphi$$

$$\cos.\ y_1Oy = \cos.\ \theta \cos.\ \psi \cos \varphi + \sin.\ \psi \sin \varphi$$

$$\cos.\ z_1Oy = \sin.\ \theta \cos.\ \psi$$

The differential angular motions, in the time dt, about the axes Ox_1, Oy_1, Oz_1, will be $v_x dt$, $v_y dt$, and $v_z dt$. We may determine the values of these motions by applying the laws of composition of rotary motion to the rotations indicated by the increments of the angles θ, φ and ψ.

If θ and φ remain constant, the increment $d\psi$ would indicate that amount of angular motion about the axis Oz perpendicular to the plane in which this angle is measured. In the same manner $d\varphi$ would indicate angular motion about the axis Oz_1; while $d\theta$ indicates rotation about the line of nodes ON. In using these three angles, therefore, we actually refer the rotation to the three axes Oz, Oz_1, ON, of which one, Oz, is fixed in

space, another, Oz_1, is fixed in and moves with the body, and the third, ON, is shifting in respect to both.

The angular motion produced around the axes Ox_1, Oy_1, Oz_1, by these simultaneous increments of the angles φ, θ and ψ, will be equal to the sum of the products of these increments by the cosines of the angles of these axes, respectively, with the lines Oz, Oz_1 and ON.

The axis of Oz_1 for example makes the angles θ, $0°$ and $90°$ with these lines, hence the angular motion $v_z dt$ is equal (taking the sum without regard to sign) to cos. $\theta d\psi + d\varphi$.

In the same manner (adding without regard to signs),

$$v_x dt = \cos.\ x_1 Oz d\psi + \cos.\ \varphi d\theta$$

and $v_y dt = \cos.\ y_1 Oz d\psi + \cos.\ (90° + \varphi) d\theta.$

But if we consider the motion about Oz_1 indicated by $d\varphi$, positive, it is plain from the directions in which φ and ψ are laid off on the figure, that the motion cos. $\theta d\psi$ will be in the reverse direction

and negative, and since cos. θ is positive $d\psi$ must be regarded as negative, hence

$$v_z dt = d\varphi - \cos.\ \theta d\psi$$

The first term of the value of $v_x dt$, cos. $x_1Ozd\psi$ [since cos. x_1Oz ($= -\sin.\theta \sin.\ \varphi$) is negative and $d\psi$ is to be taken with the negative sign] is positive. But a study of the figure will show that the rotation referred to the axis Ox_1, indicated by the first term of this value, is the reverse of that measured by a positive increment of θ in the second, and hence, (as cos. φ is positive,) $d\theta$ must be considered negative. Making this change and substituting the values given of cos. x_1Oz, cos. y_1Oz, and for cos. $(90° + \varphi)$, $-\sin.\ \varphi$, we have the three equations

$$\left.\begin{aligned} v_x dt &= \sin.\ \theta \sin.\ \varphi d\psi - \cos.\ \varphi d\theta \\ v_y dt &= \sin.\ \theta \cos.\ \varphi d\psi + \sin.\ \varphi d\theta \\ v_z dt &= d\varphi - \cos.\ \theta d\psi \end{aligned}\right\} \quad (2)^*$$

* To avoid the introduction of numerous quantities foreign to our particular investigation and a tedious analysis, I have departed from Poisson and substituted the above simple method of getting equations (2.), which is an instructive illustration of the principles of the composition of rotary motions.

The general equations (1) are susceptible of integration only in a few particular cases. Among these cases is that we consider, viz., that of a *solid of revolution* retained by a fixed point *in its axis of figure.*

Let the solid ABCD, Fig. 1, be supposed such a solid, of which Oz_1 is the axis of figure. It will be, of course, a principal axis, and any two rectangular axes in the plane, through O perpendicular to it, will likewise be principal. By way of determining them, let Ox_1 be supposed to pierce the surface in some arbitrarily assumed E point in this plane. Let G be the center of gravity (gravity being the sole accelerating force). The moments of inertia A and B become equal, and equations (1) reduce to

$$\left.\begin{array}{l} C dv_z = O \\ A dv_y - (C-A) v_z v_x dt = \gamma a M g dt \\ A dv_x + (C-A) v_y v_z dt = -\gamma b M g dt \end{array}\right\} \quad (3)^*$$

* See Bartlett's Mech. Equations (225) and (118) for the values of $L_1 M_1 N_1$; in the case we consider the extraneous force P (of eq. 118) is g; the co-ordinates $x', y,$ of its point of application G (referred to the axes Ox_1 Oy_1, Oz_1,) are zero and $z^1 = OG = \gamma$: cosines of α, β and γ are a, b and c; hence $L_1 = O$, $M_1 \gamma a M g$, $N_1 = -\gamma b M g$.

in which the distance OG of the point of support from the center of gravity is represented by γ, g is the force of gravity, M the mass and a and b stand for the cosines x_1Oz and y_1Oz and of which the values are

$$a = -\sin.\ \theta \sin.\ \varphi, \qquad b = -\sin.\ \theta \cos.\ \varphi.$$

The first equation (3) gives by integration $v_z = n$, n being an arbitrary constant; it indicates that the rotation about the axis of figure remains always constant.

Multiplying the two last equations (3) by v_y and v_x respectively and adding the products, we get

$$A(v_y dv_y + v_x\, dv_x) = \gamma Mg(av_y - bv_x)dt.$$

From the values of a and b above, and from those v_x and v_y (equations 2) it is easy to find

$$(av_y - bv_x)dt = -\sin.\ \theta\ d\,\theta = d.\cos.\ \theta;$$

substituting this value and integrating and calling h the arbitrary constant

$$A(v_y^2 + v_x^2) = 2\gamma Mg \cos.\ \theta + h. \qquad (a)$$

Multiplying the two last equations (3), respectively, by b and a and adding and reducing by the value just found of $d.$ cos. θ and of v_z, we get

$$A(bdv_y + adv_x) + (C-A)nd. \cos. \theta = O \quad (b)$$

Differentiating the values of a and b and referring to equations (2) it may readily be verified (putting for v_z its value n) that

$$db = (v_x \cos. \theta - an)dt$$
$$da = (bn - v_y \cos. \theta)dt$$

and multiplying the first by Av_y and the second by Av_x, and adding

$$A(v_y db + v_x\, da) = An(bv_x - av_y)dt$$
$$= -And. \cos. \theta.$$

Adding this to equation (b), we get

$Ad.(bv_y + av_x) + Cnd. \cos. \theta = O$, the integral of which is

$A(bv_y + av_x) + Cn \cos. \theta = l$ (l being an arbitrary constant). (c)

Referring to equations (2) it will be found by performing the operations indicated that :

$$v_x{}^2+v_y{}^2=\sin.^2\theta\frac{d\psi^2}{dt^2}+\frac{d\theta^2}{dt^2}$$

$$bv_y+av_x=-\sin.^2\theta\frac{d\psi}{dt}$$

Substituting these values in equations (a) and (c), we get

$$Cn.\cos.\theta-A\sin.^2\theta\frac{d\psi}{dt}=l$$

$$A\left(\sin.^2\theta\frac{d\psi^2}{dt^2}+\frac{d\theta^2}{dt^2}\right)=2Mg\gamma\cos.\theta+h$$

If, at the origin of motion, the axis of figure is simply deviated from a vertical position by an arbitrary angle a, in the plane of xz, and an arbitrary velocity n is imparted about this axis alone; then v_x and v_y will at that instant be zero, $\theta=a$, and the substitution of these values in equations (a) and (c) will determine the values of the constants l and h.

$$h=-2Mg\gamma\cos.a$$

$$l=Cn\cos.a,$$

which, substituted in the above equations, make them

$$\left.\begin{array}{l}\sin.^2\theta\dfrac{d\psi}{dt}=\dfrac{\mathrm{C}n}{\mathrm{A}}(\cos.\ \theta-\cos.\ a) \\ \sin.^2\theta\dfrac{d\psi^2}{dt^2}+\dfrac{d\theta^2}{dt^2}=\dfrac{2\mathrm{M}g\gamma}{\mathrm{A}} \\ \qquad\qquad\qquad\qquad(\cos.\ \theta-\cos.\ a)\end{array}\right\} \quad (4)$$

These together with the last equation (2) which may be written, (substituting the value of v_z)

$$d\varphi=ndt+\cos.\ \theta d\psi \qquad (5)$$

will (if integrated) determine the three angles φ, θ and ψ in terms of the time t. They are therefore the differential equations of motion of the gyroscope.

Let NEE′, (Fig. 1,) be a section of the solid by the plane x_1 y_1. This section may be called the *equator*. E being some fixed point in the equator (through which the principal axis Ox_1 passes), the angle φ is the angle EON.

If N is the *ascending node* of the equator—that is, the point at which E in its axial rotation *rises above* the horizontal plane, the angle φ must increase from N

towards E—that is, $d\varphi$ (in equation 5) must be positive and (as the second term of its value is usually very small compared to the first) the angular velocity n must be positive. That being the case the value of $d\varphi$ will be exactly that due to the constant axial rotation ndt, augmented by the term cos. $\theta d\psi$, which is the projection on the plane of the equator of the angular motion $d\psi$ of the node. This term is an increment to ndt when it is positive, and the reverse when it is negative. In the first case, the motion of the node is considered *retrograde*—in the second, *direct*.

The first member of the second equation (4) being essentially positive, the difference cos. θ—cos. α must be always positive—that is, the axis of figure Oz, can never rise *above* its initial angle of elevation α. As a consequence $\frac{d\psi}{dt}$ [in first equation (4)] must be always positive. The node N, therefore, moves always in the direction in which ψ is laid off positively, and the motion will be direct

or retrograde, with reference to the axial rotation, according as cos. θ is negative or positive—that is, as the axis of figure is above or below the horizontal plane. In either case the motion of the node in its own horizontal plane is always progressive in the same direction. If the rotation n were reversed, so would also be the motion of the node.

If this rotation n is zero, $\frac{d\psi}{dt}$ must also be zero and the second equation (4) reduces at once to the equation of the compound pendulum, as it should. Eliminating $\frac{d\psi}{dt}$ between the two equations (4) we get

$$\sin.^2\theta\frac{d\theta^2}{dt^2}=\frac{2Mg\gamma}{A}\left[\sin.^2\theta-\frac{C^2n^2}{2AM\gamma g}(\cos.\theta-\cos.a)\right](\cos.\theta-\cos.a).$$

The length of the simple pendulum which would make its oscillations in the same time as the body (if the rotary

velocity n were zero) is $\frac{A}{M\gamma}$.* If we call this λ and make for simplicity $\frac{C^2n^2}{2A^2g}=\frac{2\beta^2}{\lambda}$ the above equation becomes

$$\sin.^2\theta\frac{d\theta^2}{dt^2}=\frac{2g}{\lambda}[\sin.^2\theta-2\beta^2(\cos.\theta-\cos.\alpha)$$
$$(\cos.\,\theta-\cos.\,\alpha)\quad(6)$$

and the first equation (4) becomes

$$\sin.^2\theta\frac{d\psi}{dt}=2\beta\sqrt{\frac{g}{\lambda}}(\cos.\theta-\cos.\alpha\ (7)$$

Equation (6) would, if integrated, give the value of θ in terms of the time; that is, the inclination which the axis of figure makes at any moment with the vertical; while eq. (7) (after substituting the ascertained value of θ) would give the value of ψ and hence determines the progressive movement of the body about the vertical Oz.

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* The length of the simple pendulum is (see Bartlett's Mech., p. 252) $\lambda=\frac{k_1^2+\gamma^2}{\gamma}$. The moment of inertia $A = M(k_1^2+\gamma^2)$; hence $\frac{A}{M\gamma}=\lambda$.

These equations in the above general form, have not been integrated;* nevertheless they furnish the means of obtaining all that we desire with regard to gyroscopic motion, and in particular that self-sustaining power, which it is the particular object of our analysis to explain.

In the first place, from eq. (6), by putting $\frac{d\theta}{dt}$ equal to zero, we can obtain the maximum and minimum values of θ. This diff. coefficient is zero, when the factor $\cos.\ \theta - \cos.\ \alpha = 0$, that is, when $\theta = \alpha$; and this is a *maximum*, for it has just been shown from equations (4) that θ cannot exceed α. It will be zero also and θ a *minimum*† when

$$\sin.^2\theta - 2\beta^2(\cos.\ \theta - \cos.\ \alpha) = 0$$

or $$\cos.\theta = -\beta^2 + \sqrt{1 + 2\beta^2\cos.\ \alpha + \beta^4} \quad (8)$$

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* The integration may be effected by the use of elliptic functions; but the process is of no interest in this discussion.

† It is easy to show that this value of θ belongs to an actual minimum; but it is scarcely worth while to introduce the proof.

(The positive sign of the radical alone applies to the case, since the negative one would make θ a greater angle than α)

It is clear that (α being given) the value of θ depends on β alone, and that it can never become zero unless β is zero; and as long as the impressed rotary velocity n is not itself zero (however minute it may be), β will have a finite value.

Thus, however minute may be the velocity of rotation, it is sufficient to prevent the axis of rotation *from falling to a vertical position.*

The self-sustaining power of the gyroscope when very great velocities are given *is but an extreme case of this law.* For, if β is very great, the small quantity $1-\cos.^2\alpha$ may be subtracted from the quantity under the radical (eq. 8) without sensibly altering its value, which would cause that equation to become

$$\cos.\,\theta=\cos.\,\alpha$$

That is, when the impressed velocity n, and in consequence β is very great, the minimum value of θ differs from its max-

imum a by an exceedingly minute quantity.

Here then is the result, analytically found, which so surprises the observer, and for which an explanation has been so much sought and so variously given. The revolving body, though solicited by gravity, *does not visibly fall.*

Knowing this fact, we may *assume* that the impressed velocity n is very great, and hence cos. θ—cos. a exceedingly minute, and on this supposition, obtain integrals of equations (6) and (7), which will express with all requisite accuracy the true gyroscopic motion. For this purpose, make

$$\theta = a - u, \qquad d\theta = -du$$

in which the new variable u is always extremely minute, and is the angular descent of the axis of figure below its initial elevation.

By developing and neglecting the powers of u superior to the square, we have

$$\sin.^2\theta=\sin.^2\alpha-u\ \sin.\ 2\alpha+u^2\ \cos.\ 2\alpha^*$$

$$\cos.\ \theta-\cos.\ \alpha=u\ \sin.\ \alpha-\tfrac{1}{2}u^2\ \cos.\ \alpha$$

substituting these values in eq. 6, we get

$$\sqrt{\frac{g}{\lambda}}dt=\frac{du}{\sqrt{2u\ \sin.\alpha-u^2(\cos.\ \alpha+4\beta^2)}}.\dagger$$

β having been assumed very great, cos. α

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* By Stirling's theorem,

$$f(u)=U+U'\frac{u}{1}+U''\frac{u^2}{1.2},\ \&c.,$$

in which U, U ,U'', &c , are the values of $f(u)$ and its different coefficients when u is is made zero.

Making $f(u)=\sin.^2(\alpha-u)$, and recollecting that sin. $2u=2\sin.u\cos.u$ and cos. $2u=\cos.^2u-\sin^2u$, we get the value of $\sin.^2\theta$; and making $f(u)=\cos.(\alpha-u)-\cos.\alpha$ the value in text of cos. θ—cos. α is obtained.

† Eq. 6 may be written

$$\frac{\lambda}{g}\frac{d\theta^2}{dt^2}=2(\cos.\ \theta-\cos.\ \alpha)-4\beta^2\frac{(\cos.\ \theta-\cos.\ \alpha)^2}{\sin.^2\theta}.$$

By substituting the values just found of $d\theta$, $\sin.^2\theta$ and cos. θ—cos α and performing the operations indicated, neglecting the higher powers of u, (by which $\frac{(\cos.\ \theta-\cos.\ \alpha)^2}{\sin.^2\theta}$ reduces simply to u^2) and deducing the value $\sqrt{\frac{g}{\lambda}}dt$, the expression in the text is obtained.

may be neglected in comparison with $4\beta^2$ and the above may be written

$$\sqrt{\frac{g}{\lambda}}dt = \frac{du}{\sqrt{2u \sin.\, a - 4\beta^2 u^2}}. \qquad (d)$$

Integrating and observing that $u = o$, when $t = o$, we have

$$\sqrt{\frac{g}{\lambda}}.\, t = \frac{1}{2\beta}.\ \text{arc}\left\{\cos. = 1 - \frac{4\beta^2 u}{\sin. a}\right\}$$

(*See Appendix, Note A.*)

$$u = \frac{\sin. a}{4\beta^2}\left(1 - \cos.\, 2\beta\sqrt{\frac{g}{\lambda}.t}\right)$$

or, (since $\cos.\, 2a = 1 - 2 \sin.^2 a$)

$$u = \frac{1}{2\beta^2} \sin.\, a \sin.^2 \beta\sqrt{\frac{g}{\lambda}.\, t} \qquad (9)$$

Putting $a - u$ in place of θ (equat. 7) neglecting square of u, we get

$$\frac{d\psi}{dt} = \frac{1}{\beta}\sqrt{\frac{g}{\lambda}}.\ \sin.^2 \beta\sqrt{\frac{g}{\lambda}.t} \qquad (10)$$

(*See Appendix, Note B.*)

from which, observing that $\psi=0$, when $t=0$

$$\psi=\frac{1}{2\beta}\sqrt{\frac{g}{\lambda}}.t-\frac{1}{4\beta^2}\sin.\left(2\beta\sqrt{\frac{g}{\lambda}}.t\right) \qquad (11)$$

These three expressions (9), (10), (11), represent the vertical angular depression—the horizontal angular velocity—and the extent of horizontal angular motion of the axis of figure after any time t.*

The first two will reach their respective maxima and minima when $\sin.\ \beta\sqrt{\frac{g}{\lambda}}\,t$ $=1$ and $=0$; or when $t=\frac{\pi}{2\beta}\sqrt{\frac{\lambda}{g}}$ and $t=\frac{\pi}{\beta}\sqrt{\frac{\lambda}{g}}$.

These values of t in equation (11) give

$$\psi=\frac{\pi}{4\beta^2} \qquad\qquad \psi=\frac{\pi}{2\beta^2}$$

* The assumption that $\psi=0$ when t is zero supposes that the initial position of the node coincides with the fixed axis of x. In my subsequent illustrations and analysis I suppose the initial position to be at 90° therefrom, which would require to the above value of ϕ, the constant $\frac{1}{2}\pi$ to be added. The horizontal angular motion of the axis of figure is the same as that of the node.

Hence, counting from the commencement of motion, when t, u, $\frac{d\psi}{dt}$ and ψ are all zero, we have the following series of corresponding values of these variables

$$t=\frac{\pi}{2\beta}\sqrt{\frac{\lambda}{g}},\quad u=\frac{1}{2\beta^2}\sin.\ a\ \frac{d\psi}{dt}=\frac{1}{\beta}\sqrt{\frac{g}{\lambda}}$$

$$\psi=\frac{\pi}{4\beta^2}$$

which correspond to the moment of greatest depression, when u and $\frac{d\psi}{dt}$ are maxima and

$$t=\frac{\pi}{\beta}\sqrt{\frac{\lambda}{g}},\quad u=0, \frac{d\psi}{dt}=0,\ \psi=\frac{\pi}{2\beta^2},$$

when, it appears (u being the zero), the axis of figure has regained its original elevation and the horizontal velocity is destroyed.

All these values are (owing to the assumed large value of β) very minute. If we suppose the rotating velocity $n=100$ π or 100 revolutions per second, the maximum of u (with an instrument of or-

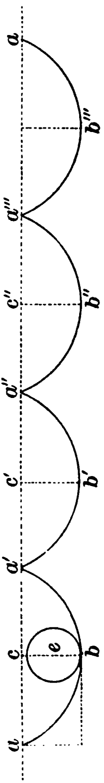
dinary proportions) would be a fraction of a minute of arc, and the period of undulation but a fraction of a second.

Hence the horizontal motion about the point of support will be exceedingly slow compared with the axial rotation of the disk expressed by n.

If, in equations (9) and (10), we increase t indefinitely we will have but a repetition of the series of values already found, they being recurring functions of the time.

We see, then, the revolving body *does not*, in fact, maintain a uniform unchanging elevation, and move about its point of support at a uniform rate, (as it appears to do). But the axis of figure generates what may be called a *corrugated cone*, and any point of it would describe an undulating curve (Fig. 2) whose superior culminations a, a', a'', &c., are *cusps* lying in the same horizontal plane, and whose sagittae cb, $c'b'$,&c., are to the amplitudes aa', a',a'' &c., as $\frac{\sin. a}{2\beta^2} : \frac{\pi}{2\beta^2} : :$ $\sin. a : \pi$. If the initial elevation a is 90°,

Fig. 2



this ratio is *as the diameter to the circumference of the circle:* a property which indicates the *cycloid.*

Assuming $\alpha=90°$ and sin. $\alpha=1$, equations (9) and (10) will give, by elimination of $\sin.^2\beta\sqrt{\frac{g}{\lambda}}t$,

$$\frac{d\psi}{dt}=2\beta\sqrt{\frac{g}{\lambda}}u \qquad dt=\frac{d\psi}{2\beta\sqrt{\frac{g}{\lambda}}u},$$

substituting this value in eq. (d), we get

$$d\psi=\frac{2\beta\, u\, d\, u}{\sqrt{2u-4\beta^2u^2}}=\frac{u\, d\, u}{\sqrt{\frac{1}{2\beta^2}u-u^2}}$$

the differential equation of the cycloid generated by the circle whose diameter is $\frac{1}{2\beta^2}$

In this position of the axis, both the angles u and ψ are arcs of great circles described by a point of the axis of figure at a units distance from O, and owing to

their minnteness may be considered as rectilinear co-ordinates.

If α is not 90°, the sagittae $bc = \frac{1}{2\beta^2}$ sin. α; but then, while the angular motion ψ is the same, the arc described by the same point of the axis will be that of a *small circle*, whose actual length will likewise be reduced in the ratio of 1 : sin. α. The curve is therefore a cycloid in all circumstances; and the axis of figure moves as if it were attached to the circumference of a minute circle whose diameter is $\frac{1}{2\beta^2}$ sin. α, which rolled along the horizontal circle, a a' a'', about the vertical through the point of support.

The center e of this little circle moves with uniform velocity. The *first term* of the value of ψ (equation 11) is due to this uniform motion; it may be called the *mean precession.*

The second term is due to the circular motion of the axis about this center, and combined with the corresponding values

of u, constitutes what may be called the *nutation.*

These cycloidal undulations are so minute—succeed each other with such rapidity (with the high degreees of velocity usually given to the gyroscope), that they are entirely lost to the eye, and the axis seems to maintain an unvarying elevation and move around the vertical with a uniform slow motion.

It is in omitting to take into account these minute undulations that nearly all popular explanations fail. They fail, in the first place, because they substitute, in the place of the real phenomenon, one which is purely imaginary and *inexplicable*, since it is in direct variance with fact and the laws of nature;—and they fail, because these undulations—(great or small, according as the impressed rotation is small or great) furnish the only true clue to an understanding of the subject.

The fact is, that the phenomenon exhibited by the gyroscope which is so striking, and for which explanations are

so much sought, is only a *particular and extreme phase* of the motion expressed by equations (6) and (7)—that the self-sustaining power is not absolute, but one of degree—that, however minute the axial rotation may be, the body never will fall quite to the vertical;—however great, it cannot sustain itself without any depression.

I have exhibited the undulations, as they exist with high velocities—when they become minute and nearly true cycloids; with low velocities they would occupy (horizontally) a larger portion of the arc of a semi-circle, and reach downward approximating, more or less nearly, to contact with the vertical; and, *finally*, when the rotary velocity is zero, their cusps are in diametrically opposite points of the horizontal circle, while the curves resolve themselves into vertical circular arcs which coincide with each other, and the vibration of the pendulum is exhibited. All these varieties of motion, of which that of the pendulum is one extreme phase and the gyroscopic another, are

embraced in equations (6) and (7) and exhibited by varying β from 0 to high values, though (wanting general integrals to these equations) we cannot determine, except in these extreme cases, the exact elements of the undulations. The minimum value of θ may, however, always be determined by equation (8).

If we scrutinize the *meaning* of equations (6) and (7), it will be found that they represent, the first, the horizontal angular component of the velocity of a point at units distance from O, and the second the actual velocity of such point.*

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*In more general terms equations(4) express.the first, that the *moment of the quantity of motion* about the fixed vertical axis Oz remains always constant; the *second* that the living forces generated in the body (over and above the *impressed* axial rotation) are exactly what is *due to gravity through the height, h.*

Both are expressions of truths that might have been anticipated; for gravity cannot increase the moment of the quantity of motion about an *axis parallel to itself;* while its power of generating living force by working through a given height, cannot be impaired.

Had we considered ourselves at liberty to assume them, however, the equations might have been got without the tedious analysis by which we have reached them.

For $\sin.\theta\frac{d\psi}{dt}$ is the horizontal, and $\frac{d\theta}{dt}$ the vertical, component of this velocity. Calling the first v_h, and the second v_v, and the resultant v_s, and calling cos. θ—cos. a (which is the true height of fall), h, those equations may be written

$$v_h = \frac{Cn}{A}\,\frac{h}{\sin.\,\theta} \qquad (e)$$

$$(v_h^2 + v_v^2) = v_s^2 = \frac{2g}{\lambda}h \qquad (f)$$

This velocity v_s (as a function of the height of fall) is exactly that of *the compound pendulum*, and is *entirely independent of the axial rotation n.* Hence (as we might reasonably suppose) rotary motion has no power to impair the work of gravity *through a given height*, in generating velocity; but it does have power to *change the direction of that velocity.* Its effect is precisely that of a material undulatory curve, which, deflecting the body's path from vertical descent, finally

directs it upward, and causes its velocity to be destroyed by the same forces which generated it.

And it may be remarked, that, were the cycloid we have described *such a material curve*, on which the axis of the gyroscope rested, without friction and *without rotation*, it would travel along this curve by the effect of gravity alone (the velocity of descent on the downward branch carrying it up the ascending one), with *exactly the same velocity* that the rotating disk does, through the combined effects of gravity *and* rotation.

Equation (a) expresses the horizontal velocity produced by the rotation.

If we substitute its value in the second we may deduce

$$v_v \text{ or } \frac{d\theta}{dt} = \sqrt{\frac{2g}{\lambda}h - \frac{C^2n^2}{A^2}\,\frac{h^2}{\sin.^2\theta}}$$

If we take this value at the commencement of descent, *and before any horizontal velocity is acquired* (making h indefinitely small), the second term under the radical may be neglected, and the

first increment of descending velocity becomes $\sqrt{\frac{2g}{\lambda}h}$, precisely what is due to gravity, and *what it would be were there no rotation.*

Hence the popular idea that a rotating body offers any *direct* resistance to a change of its plane, is unfounded. It requires as little exertion of force (in the direction of motion) to move it from one plane to another, as if no rotation existed; and(as a corollary) as little expenditure of work.

But deflecting forces are developed, by angular motion given to the axis, and normal to its direction, which are very sensible, and are mistaken for *direct* resistances. If the extremity of the axis of rotation were confined in a vertical circular groove, in which it could move without friction; or if any similar fixed resistance, as a material vertical plane, were opposed to the *deflecting* force, the rotating disk would vibrate in the vertical plane as if no rotation existed. Its equation of motion would become that of the

compound pendulum, $\frac{d\theta}{dt}=\sqrt{\frac{2g}{\lambda}h}$. What then is the resistance to a change of plane of rotation so often alluded to and described. A *misnomer* entirely.

The above may be otherwise established. If in equations (3) we introduce in the second member an indeterminate horizontal force, g', applied to the center of gravity, parallel to the fixed axis of y, and contrary to the direction in which, in our figure, we suppose the angle ψ to increase, the projections of this force on the axes Ox_1, Oy_1, will be $a'\ g'$ and $b'\ g'$ and the last two of these equations will become (calling cosines $x_1\ Oy$ and y_1Oy_1 a$'$ and b',)

$$Adv_y—(C—A)nv_x\,dt=\gamma M(ag+a'g')dt$$
$$Adv_x+(C-A)nv_y dt=—\gamma M(bg+b'g')dt$$

Multiplying the first by v_y and the second by v_x and adding

$$A(v_y dv_y+v^x\,dv_x)=\gamma M[g(av_y-bv_x)dt+g'(a'v_y-b'v_x)dt]$$

But $(av_y - bv_x)dt$ has been shown to be $= d. \cos. \theta$,—and by a similar process it may be shown that $(a'v_y - b'v_x)dt = d.$ $(\sin. \theta \cos. \psi)$. (For values of a' and b' as before.)

Let us suppose now that the force g' is such that the axis of the disk may be always maintained in the plane of its initial position xz. The angle ψ would always be 90°, $d\psi = 0$, and $d.(\sin. \theta \cos. \psi) = 0$. That is, the co-efficient of the new force g' becomes zero; and the integral of the above equation is as before,

$$A(v_y^2 + v_x^2) = 2\gamma M g \cos. \theta + h.$$

But the value of $v_y^2 + v_x^2$ likewise reduces (since $\frac{d\psi}{dt} = 0$) to $\frac{d\theta^2}{dt^2}$ and the above becomes the equation of the compound pendulum (g) $\frac{d\theta^2}{dt^2} = \frac{2\gamma M g}{A} \cos. \theta + h = \frac{2g}{\lambda}$ $(\cos. \theta - \cos. a)$, (h being determined.) This is the principle just before announced, that, with a force so applied as to prevent any *deflection* from the plane in which gravity tends to cause the axis to

vibrate, the motion would be precisely *as if no axial rotation existed.*

To determine the force of g'; multiply the first of preceding equations by b, and the second by a, and add the two, and add likewise $A(v_y db + v_x da = -A\, nd \cos.\, \theta$, and we shall get

$$Ad(bv_y + av_x) + C\, nd \cos.\, \theta = \gamma M g' (a'b - ab')dt.$$

By referring to the values of a, a', b, b', and performing the operations indicated and making $\cos.\, \psi = o$, $\sin.\, \psi = 1$, the above becomes,

$$Ad(bv_y + av_x) + C\, nd \cos.\, \theta = \gamma M g' \sin.\, \theta\, dt.$$

But the value of $(bv_y + av_x)$ becomes zero when $\frac{d\psi}{dt} = o$. Hence,

$$g' = \frac{C\, nd \cos.\, \theta}{\gamma M \sin.\, \theta\, dt} = -\frac{Cn\, d\theta}{\gamma M\, dt}*$$

---

* The effect of gravity is to diminish θ and the increment $d\theta$ is negative in the case we are considering. Hence the negative sign to the value of g', indicating

The second factor $\frac{d\theta}{dt}$ is the *angular velocity* with which the axis of rotation is moving.

Hence calling v_s that angular velocity, *the value of the deflecting force*, g' may be written (irrespective of signs),

$$g' = \frac{C}{\gamma M} n v_s. \qquad (h)$$

that is, it is directly proportional *to the axial rotation* n, and *to the angular velocity* of the axis of that rotation. By putting for C,Mk^2 (in which k is the distance from the axis at which the mass M, if concentrated, would have the moment of inertia, C,) the above takes the simple form

$$g' = \frac{k^2}{\gamma} n v_s.$$

In the case we have been considering above, in which g' is supposed to coun-

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that the force is in the direction of the *positive* axis of y, as it should, since the tendency of the node is to move in the reverse direction.

teract the deflecting force of axial rotation, the angular velocity v_s or $-\frac{d\theta}{dt}$ (equation g) is equal to $\sqrt{\frac{2g}{\lambda}(\cos.\ \theta - \cos.\ a)}$

But in the case of the *free* motion of the gyroscope, this deflecting force combines with gravity to produce the observed movements of the axis of figure.

If, *therefore*, we *disregard* the axial rotation and consider the body simply as fixed at the point O, and acted upon, at the center of gravity, by two forces—one of gravity constant in intensity and direction—the other, the *deflecting* force due to an axial rotation n, whose variable intensity is represented by $\frac{C}{\gamma M}$ nv_s, and whose direction is always normal to the plane of motion of the axis; we ought, introducing these forces, and making the axial rotation n zero, in general equations (3), to be able to deduce therefrom the identical equations (4) which express the motion of the gyroscope.

This I have done; but as it is only a verification of what has previously been said, I omit in the text the introduction of the somewhat difficult analysis.

(*See Appendix, Note C.*)

Equation (5) becomes (in the case we consider), by integration;

$$\varphi = nt + \psi \cos. a$$

which, with the values of u and ψ already obtained, determines completely the position of the body at any instant of time.

Knowing now not only the exact nature of the motion of the gyroscope, but the direction and intensity of the forces which produce it, it is not difficult to understand why such a motion takes place.

Fig. 1 represents the body as supported by a point *within* its mass; but the analysis applies to any position, in the axis of figure, within or without; and Figs. 3 and 4 represent the more familiar cir-

Fig. 3

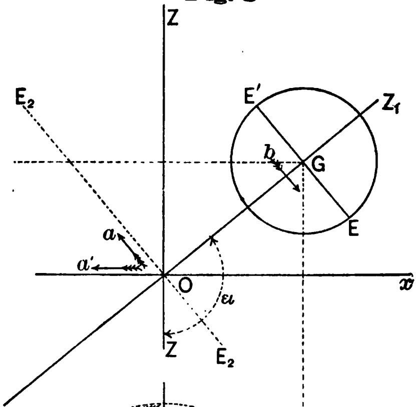
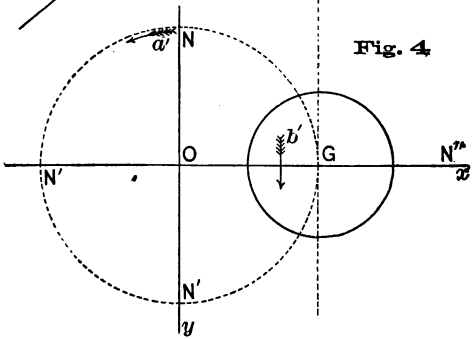


Fig. 4



cumstances under which the phenomenon is exhibited.

Let the revolving body be supposed (Fig. 3, vertical projection), for simplicity of projection, an exact *sphere*, supported by a point in the axis prolonged at O, which has an initial elevation a greater than 90°. Fig. 4. represents the projection on the horizontal plane xy the initial position of the axis of figure (being in the plane of xz) is projected in Ox.

Ox, Oy, Oz, are the three (fixed in space) co-ordinate axes, to which the body's position is referred.

In this position, an initial and high velocity n is supposed to be given about the axis of figure Oz_1, so that the visible portions move in the direction of the arrows b, b', and the body is left subject to whatever motion about its point of support O, gravity may impress upon it. Had it no axial rotation, it would immediately fall and vibrate according to the known laws of the pendulum. Instead of which, while the axis maintains

(apparently) its elevation a, it moves slowly around the vertical Oz, receding from the observer, or from the position ON'' towards ON.

It is self-evident that the first *tendency* (and as I have likewise proved, the first effect) of gravity is to cause the axis Oz_1 to descend vertically, and to generate vertical *angular velocity*. But with this angular velocity, the *deflecting* force proportional to that velocity and normal to its direction, is generated, which pushes aside the descending axis from its vertical path. But as the direction of motion changes, so does the direction of this force—always preserving its perpendicularity. It finally acquires an intensity and upward direction adequate to neutralize the downward action of gravity; but the *acquired downward velocity* still exists and the axis *still* descends, at the same time acquiring a constantly increasing horizontal component, and with it a still increasing upward deflecting force. At length the descending component of velocity is entirely destroyed—the path

of the axis is horizontal; the deflecting force due to it acts directly contrary to gravity, which it exceeds in intensity, and hence causes the axis to commence rising. This is the state of things at the point *b* (Fig. 2). The axis has descended the curve *ab*, and has acquired a velocity due to its *actual* fall *ad*; but this velocity has been deflected to a *horizontal direction.* The *ascent* of the branch *ba′* is precisely the converse of its descent. The *acquired* horizontal velocity impels the axis horizontally, while the deflecting force due to it (now at its maximum) causes it to commence ascending. As the curve bends upward, the normal direction of this force opposes itself more and more to the horizontal, while gravity is equally counteracting the vertical velocity. As the *horizontal* velocity at *b* was due to a fall through the height *ad*, so, through the medium of this deflecting force, it is just capable of restoring the *work* gravity had expended and *lifting* the axis back to its original elevation at *a′*, and the cycloidal undulation is completed, to be again and

again repeated, and the axis of figure, performing undulations too rapid and too minute to be perceived, moves slowly around its point of support.

Referring to Fig. 3, the *equator* of the revolving body (a plane perpendicular to the axis of figure and *through the fixed point* O,) would be an imaginary plane E_1E_2. Its intersection with the horizontal plane of *xy* would be the line of nodes N,N′. In the position delineated, the progression of the nodes is *direct*. For, at the *acending* node N, any point in the imaginary plane of the equator (suppoosed to revolve with the body) would move *upwards* in the direction of the arrow *a*, while the node moves in the *same direction from* O (of the arrow *a′*). Were the axis of figure below the horizontal plane, (Fig. 5) the upward rotation of the point would be from O to E_2 (as the arrow *a*), while the progression of the node (in the same direction as before as the arrow *a′*) would be the reverse, and the motion of the node would be *retro-*

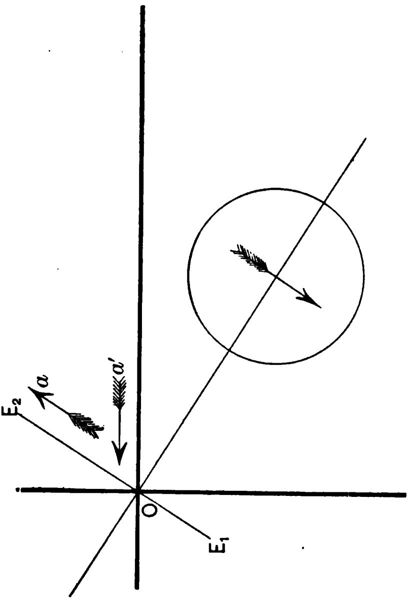


Fig. 5

grade—yet in both cases the same in space.

As the deflecting force of rotary motion is the sole agent in diverting the vertical velocity produced by gravity from its downward direction, and in producing these paradoxical effects; and as the foregoing analysis, while it has determined its value, has thrown no light upon its origin, it may be well to inquire how this force is created.

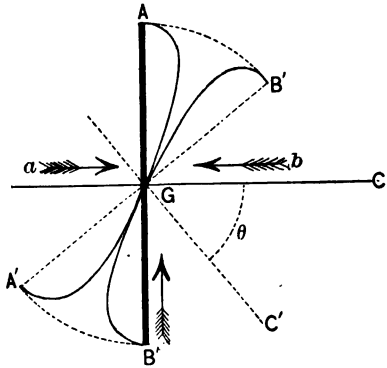
Popular explanations have usually turned upon the deflection of the *vertical* components of rotary velocity by the vertical angular motion of the axis produced by gravity. In point of fact, however *both* vertical and horizontal components are deflected, one as much as the other; and the simplest way of studying the effects produced, is to trace a vertical projection of the path of a point of the body under these combined motions. For this purpose conceive the mass of the revolving disk concentrated in a single ring of matter of a radius k due to its moment of inertia $C=Mk^2$ (see Bartlett Mech. p.

178), and, for simplicity, suppose the angular motion of the axis to take place around the center figure and of gravity G.

Let AB be such a ring (supposed perpendicular to the plane of projection), revolving about its axis of figure GC, while the axis turns *in the vertical plane* about the same point G. Let the rotation, be such that the visible portion of the disk moves upward through the semi-circumference, from B to A, while the axis moves downward through the angle θ to the position GC′. The point B, by its *axial rotation* alone, would be carried to A; but the plane of the disk, by simultaneous movement of the axis, is carried to the position A′B′, and the point B arrives at B′ instead of A, through the curve projected in BGB′. The equation of the projection, in circular functions, is easily made; but its general character is readily perceived, and it is sufficient to say that it passes through the point G,—that its tangents at B and B′ are perpendicular to AB and A′B′,—and that its concavity throughout its whole length turned to

the right. The point A descends on the other, or remote side of the disk, and makes an exactly similar curve AGA′ with its concavity reversed.

Fig. 6



The *centrifugal* forces due to the deflections of the vertical motions are normal to the concavities of these curves ; hence, on the side of the axis *towards* the eye, they are to the *left*, and on the opposite or further side, to the *right*, (as the arrows *b* and *a*.) Hence the joint effect is to

press the axis GC from its vertical plane CGC′, horizontally and towards the eye. Reverse the direction of axial rotation and the curves AA′ and BB′ will be the same except that AA′ would be on the *near* and BB′ on the *remote* side of the axis GC, and the direction of the resulting pressure will be reversed.

A projection on the horizontal plane would likewise illustrate this deflecting force and show at the same time that there is *no resistance in the plane of motion of the axis*, and that the whole effect of these deflections of the paths of the different material points, is a mere *interchange of living forces between the different material points of the disk;* but it is believed that the foregoing illustration is sufficient to explain the *origin* of this force, whose measure and direction I have analytically demonstrated.

It may be remarked, however, that the intensity of the force will evidently be directly as the velocities *gained and lost* in the motion of the particles from one

side of the axis to the other; or as the *angular velocity of the axis*, and as the distance, k, of the particles from that axis. It will also be as the *number of particles* which undergo this gain and loss of living force in a given time; or *as the velocity of axial rotation*. Considered as applied normally at G to produce rotation about *any* fixed point O in the axis, its intensity will evidently be *directly* as the arm of lever k, and *inversely* as the distance of G from O (γ). Hence the measure of this force already found, from analysis,

$$g' = \frac{k^2}{\gamma} n v_s.$$

In the foregoing analysis, the entire ponderable mass is supposed to partake of the impressed rotation about the axis of figure Oz_1; and such must be the case im order that the results we have arrived at may rigidly apply. Such, however, cannot be the case in practice. A portion of the instrument must consist of mountings which do not share in the rotation of the disk. It is believed the analysis will apply to this case by simply in-

cluding the *whole mass*, in computing the moment of inertia A and the mass M, while the moment C represents, as before, that of *the disk alone*.

In this manner it would be easy to calculate what *amount of extraneous* weight (with an *assumed* maximum depression u) the instrument would sustain, with a given velocity of rotation.

The analogy between the minute motions of the gyroscope and that grand phenomenon exhibited in the heavens,—the "precession of the equinoxes"—is often remarked. In an ultimate analysis, the phenomena, doubtless, are identical; yet the immediate causes of the latter are so much more complex, that it is difficult to institute any profitable comparison.

At first sight the undulatory motion attending the precession, known as "nutation" (nodding) would seem identical with the undulations of the gyroscope. But the identity is not easily indicated; for the earth's motion of nutation is mainly governed by the moon, with whose cycles it coincides; and the

solar and lunar precessions and nutations are so combined, and affected by causes which do not enter into our problem, that it is vain to attempt any minute identification of the phenomena, without reference to the difficult analysis of celestial mechanics.

On a preceding page I said that a horizontal motion of the rotating disk around its point of support, without descending undulations, was at variance with the laws of nature. This assertion applied, however, only to the actual problem in hand, in which no other external force than gravity was considered, and no other initial velocity than that of axial rotation.

Analysis shows, however, that an initial *impulse* may be applied to the rotating disk in such a way that the horizontal motion shall be absolutely without undulation. An initial horizontal angular velocity, such as would make its corresponding deflective force equal to the component of gravity, g sin. θ, would

cause a horizontal motion *without* undulation.

If the axial rotation n, as well as the horizontal rotation, is communicated by an impulsive force, analysis shows that it may be applied in *any plane* intersecting the horizontal plane *in the line of nodes;* but if applied in the plane of the equator (where it can communicate nothing but an *axial* rotation n), or in the horizontal plane, its intensity must be infinite.

My announced object does not carry me further into the consideration of the gyroscope than the solution of this peculiar phenomenon, which depends solely upon, and is so illustrative of, the laws of rotary motion.

If I have been at all successful in making this so often explained subject more intelligible—in giving clearer views of some of the supposed effects of rotation, it has been because I have trusted solely to the *only* safe guide in the complicated phenomena of nature, *analysis.*

APPENDIX.—NOTE A.

$\dfrac{du}{\sqrt{2u \text{ sin. } a - 4\beta^2 u^2}}$ may be put in the form $\dfrac{2\beta}{\text{sin. } a} \cdot \dfrac{\frac{\text{sin. } a}{4\beta^2} du}{\sqrt{2u \frac{\text{sin. } a}{4\beta^2} - u^2}}$

Call $\dfrac{\text{sin. } a}{4\beta^2} = \text{R}$, and the integral of the 2d factor of the above is the arc whose radius is R and versed sine is u; or whse cosine is R$-u$, or it is R *times* the arc whose cosine $1-\dfrac{u}{\text{R}}$ with radius unity. Substituting the value of R in the integral and multiplying by the factor $\dfrac{2\beta}{\text{sin. } a}$ we get the value of $\sqrt{\dfrac{g}{\lambda}}t$, of the text.

Note B.

In eq. (7) if we divide both members by $\sin.^2\theta$, and, in reducing the fraction $\frac{\cos.\ \theta-\cos.\ a}{\sin.^2\ \theta.}$, use the values already found and neglect the *square*, as well as higher powers u, (which may be done without sensible error, owing to the minuteness of u, though it could not be done in the foregoing values of dt and t, since the co-efficient $4\beta^2$ in those values, is reciprocally great, as u is small) the quotient will be simply $\frac{u}{\sin.\ a}$

Substituting the value of u and dividing out $\sin.\ a$, we get the value of $\frac{d\psi}{dt}$ in the text.

The integral of $\sin.^2\beta\sqrt{\frac{g}{\lambda}}tdt$ results from the formula $\int\sin.^2\ \varphi d\varphi=\frac{1}{2}\varphi-\frac{1}{4}\sin.\ 2\varphi$, easily obtained by substituting for $\sin.^2$ φ, its value $\frac{1}{2}-\frac{1}{2}\cos.\ 2\varphi$.

NOTE C.

To introduce these forces in Eq. (3) I observe, first, that as both are applied at G (in the axis Oz_1) the moment L_1 is still zero and the *first* eq. becomes, as before

$$Cdv_z = O \text{ or } v_z = \text{const.}$$

And as we disregard the impressed axial rotation, we make this constant (or v_z) zero.

The deflecting force $\frac{Cn}{\gamma M}v_s$ (taken with contrary sign to the *counteracting* force just obtained) resolves itself into two components $\frac{Cn}{\gamma M}\frac{d\,\theta}{dt}$ and $-\frac{Cn}{\gamma M}\frac{d\psi}{dt}$ sin. θ, the first in a horizontal, the second in a vertical plane, and both normal to the axis of figure.

The second is opposed to gravity, whose component normal to the axis of figure is g sin. θ.

Hence we have the two component forces (in the directions above indicated),

$$M.\frac{Cn}{\gamma M}\frac{d\theta}{dt} \text{ and } M\left(g - \frac{Cn}{\gamma M}\frac{d\psi}{dt}\right)\sin,\ \theta.$$

These moments with reference to the axes of y_1 and x_1 will be

$$-\sin.\ \varphi\gamma M\left(g - \frac{Cn}{\gamma M}\frac{d\psi}{dt}\right)\sin.\ \theta - \cos.\ \varphi\gamma M\frac{Cn}{\gamma M}\frac{d\theta}{dt}, \text{ and}$$

$$\cos.\ \varphi\gamma M\left(g - \frac{Cn}{\gamma M}\frac{d\psi}{dt}\right)\sin.\ \theta - \sin.\ \varphi\gamma M\frac{Cn}{\gamma M}\frac{d\theta}{dt}$$

Hence equations (3) (making v_z zero, and putting for M_1 and N_1 the above values, and recollecting the values of a and b, become

$$\left.\begin{aligned}
Adv_y &= a_\gamma Mgdt - \\
&\qquad aCn\frac{d\psi}{dt}dt - Cn\cos.\,\varphi\frac{d\,\theta}{dt}dt \\
Adv_x &= -b_\gamma Mgdt + \\
&\qquad bCn\frac{d\psi}{dt}dt - Cn\sin.\,\varphi\frac{d\,\theta}{dt}\,dt
\end{aligned}\right\} i$$

Multiplying the equations severally by v_y and v_x, adding and reducing, we get

$$A(v_y\,dv_y + v_x\,dv_x) = \gamma Mgd.\cos.\,\theta -$$
$$Cn\frac{d\psi}{dt}d.\cos.\theta - Cnd\,\theta\,(v_y\cos.\psi + v_x\sin.\varphi)$$

But $v_y\cos.\,\varphi + v_x\sin.\,\varphi$ will be found equal to $\sin.\,\theta\frac{d\psi}{dt}$ (by substituting the values of v_y and v_x); hence the two last terms destroy each other, and the above equation becomes identical with equation (a) from which the 2d eq. (4) is deduced.

Multiplying the 1st equation (i) by $\cos.\,\varphi$ and the second by $\sin.\,\varphi$ and adding, we get

$$A(\cos.\,\varphi dv_y + \sin.\,\varphi dv_x) = -Cnd\,\theta.$$

By differentiating the values of v_y and v_x, performing the multiplications, and substituting for $d\varphi$ its value, cos. $\theta d\psi$, (proceeding from the 3d equation (2) when $v_z = 0$), the above becomes

$$A\left(\sin.\ \theta \frac{d^2\psi}{dt^2} + 2\cos.\ \theta \frac{d\psi}{dt}\frac{d\theta}{dt}\right) = -Cn\frac{d\theta}{dt}.$$

Multiplying both members by sin. θdt, and integrating, the above becomes

$$\sin.^2\ \theta \frac{d\psi}{dt} = \frac{Cn}{A}\cos.\ \theta + l;$$

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